අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2022(2023)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2022(2023)
General Certificate of Education (Adv. Level) Examination, 2022(2023)

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

Part B

※ Answer **five** questions only.

11. (*a*) Let $0<|p|<1$. Show that the equation $p^2x^2-2x+1=0$ has real distinct roots.
Let α and β $(>\alpha)$ be these roots. Show that α and β are both positive.
Find $(\alpha-1)(\beta-1)$ in terms of p, and **deduce** that $\alpha<1$ and $\beta>1$.
Show that $\sqrt{\beta}-\sqrt{\alpha}=\frac{1}{|p|}\sqrt{2(1-|p|)}$

It is **given that** $\sqrt{\beta}+\sqrt{\alpha}=\frac{1}{|p|}\sqrt{2(1+|p|)}$. Show that the quadratic equation whose roots are $\left|\sqrt{\alpha}-1\right|$ and $\left|\sqrt{\beta}-1\right|$ is $|p|x^2-\sqrt{2(1-|p|)}\,x+\sqrt{2(1+|p|)}-|p|-1=0$.

(*b*) Let $p(x)=2x^3+ax^2+bx-4$, where $a, b\in\mathbb{R}$. It is given that $(x+2)$ is a factor of both $p(x)$ and $p'(x)$, where $p'(x)$ is the derivative of $p(x)$ with respect to x. Find the values of a and b. For these values of a and b, completely factorise $p(x)-3p'(x)$.

12. (*a*) Six mangoes and four oranges are to be distributed among eight students so that each student receives at least one fruit.
Find the number of different ways in which
(i) six students get one fruit each and out of the remaining two students one gets **two mangoes** and the other gets **two oranges**,
(ii) seven students get one fruit each, and the other student gets **three mangoes**,
(iii) seven students get one fruit each, and the other student gets **three fruits**.

(*b*) Let $U_r=\frac{4(2r+7)}{(2r+1)(2r+3)(2r+5)}$ for $r\in\mathbb{Z}^+$. Also, let $f(r)=\frac{A}{(2r+1)}+\frac{B}{(2r+3)}$ for $r\in\mathbb{Z}^+$, where A and B are real constants. Determine the values of A and B such that $U_r=f(r)-f(r+1)$ for $r\in\mathbb{Z}^+$.

Hence or otherwise, show that $\sum_{r=1}^{n}U_r=\frac{4}{5}-\frac{3}{2n+3}+\frac{1}{2n+5}$ for $n\in\mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty}U_r$ is convergent and find its sum.

Hence, find the value of the real constant k such that $\sum_{r=1}^{\infty}(U_r+kU_{r+1})=1$.

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13.(*a*) Let $\mathbf{A} = \begin{pmatrix} a & -2 \\ 1 & a+2 \end{pmatrix}$. Show that $\mathbf{A}^{-1}$ exists for all $a \in \mathbb{R}$.

The matrices $\mathbf{P} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 2 \\ -1 & 7 & 4 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ are such that $\mathbf{A} = \mathbf{P}\mathbf{Q}^{\mathrm{T}} + \mathbf{R}$. Show that $a = 1$.

For this value of a, write down $\mathbf{A}^{-1}$ and **hence**, find the values of x and y such that $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$.

(*b*) Let $z, w \in \mathbb{C}$. Show that $z\bar{z} = |z|^2$ and **hence**, show that $|z+w|^2 = |z|^2 + 2\,\mathrm{Re}(z\bar{w}) + |w|^2$.

Deduce that $|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$ and give a geometric interpretation for it when the points representing z, w and 0 in the Argand diagram are non-collinear.

(*c*) Let $z = -1 + \sqrt{3}i$. Express z in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\frac{\pi}{2} < \theta < \pi$.

Let $z^n = a_n + ib_n$, where $a_n, b_n \in \mathbb{R}$ for $n \in \mathbb{Z}^+$. Write down $\mathrm{Re}\left(z^m \cdot z^n\right)$ in terms of a_m, a_n, b_m and b_n for $m, n \in \mathbb{Z}^+$.

Considering z^{m+n} and using De Moivre's theorem, show that $a_m a_n - b_m b_n = 2^{m+n}\cos(m+n)\frac{2\pi}{3}$, for $m, n \in \mathbb{Z}^+$.

14.(*a*) Let $f(x) = \dfrac{2x+3}{(x+2)^2}$ for $x \neq -2$.

Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{-2(x+1)}{(x+2)^3}$ for $x \neq -2$.

Hence, find the interval on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Also, find the coordinates of the turning point of $f(x)$.

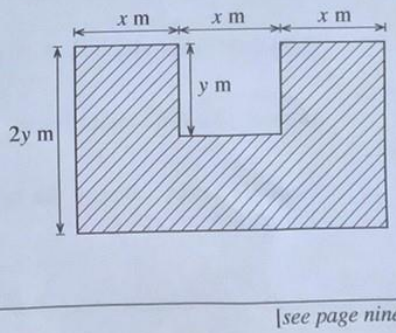
It is given that $f''(x) = \dfrac{2(2x+1)}{(x+2)^4}$ for $x \neq -2$. Find the coordinates of the point of inflection of the graph of $y = f(x)$.

Sketch the graph of $y = f(x)$ indicating the asymptotes, the turning point and the point of inflection.

State the smallest value of k for which $f(x)$ is one-one on $[k, \infty)$.

(*b*) The shaded region shown in the figure is of area 45 m^2. It is obtained by removing a rectangle of length x m and width y m from a rectangle of length $3x$ m and width $2y$ m. Show that the perimeter L m of the shaded region is given by $L = 6x + \dfrac{54}{x}$ for $x > 0$.

Find the value of x such that L is minimum.



[see page nine

15. (*a*) Find the values of the constants A, B and C such that

$x^2+x+2=A(x^2+x+1)+(Bx+C)(x+1)$ for all $x\in\mathbb{R}$.

Hence, write down $\dfrac{x^2+x+2}{(x^2+x+1)(x+1)}$ in partial fractions and find $\displaystyle\int\frac{x^2+x+2}{(x^2+x+1)(x+1)}\,dx$.

(*b*) Show that $1+\sin 2x=2\cos^2\left(\frac{\pi}{4}-x\right)$ and **hence**, show that $\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin 2x}\,dx=1$.

(*c*) Let $I=\displaystyle\int_0^{\frac{\pi}{2}}\frac{x^2\cos 2x}{(1+\sin 2x)^2}\,dx$. Using integration by parts, show that $I=-\frac{\pi^2}{8}+J$, where $J=\displaystyle\int_0^{\frac{\pi}{2}}\frac{x}{1+\sin 2x}\,dx$.

Using the relation $\displaystyle\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$ and the result in (*b*), evaluate J and show that $I=\frac{\pi}{8}(2-\pi)$.

16. Let $P\equiv(x_0,y_0)$ and l be the straight line given by $ax+by+c=0$. Show that the perpendicular distance from P to l is $\dfrac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.

Let l_1 and l_2 be two straight lines given by $4x-3y+8=0$ and $3x-4y+13=0$, respectively. Show that l_1 and l_2 intersect at $A\equiv(1,\,4)$.

Also, show that the parametric equations of the bisector of the acute angle between l_1 and l_2 can be written as $x=t$ and $y=t+3$, where $t\in\mathbb{R}$.

Hence, show that the equation of any circle touching both straight lines l_1 and l_2, and lying in the region between l_1 and l_2 that contains the acute angle, is given by $(x-t)^2+(y-t-3)^2=\frac{1}{25}(t-1)^2$, where $t\in\mathbb{R}$ and $t\neq 1$.

From among the above circles, find the equations of the circles that intersect the circle centred at A of radius 1, orthogonally.

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17. (*a*) Write down $\cos(A+B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$, and obtain a similar expression for $\sin(A-B)$.

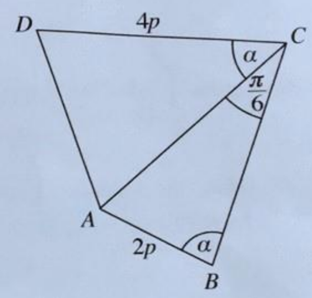
Let $k \in \mathbb{R}$ and $k \neq 1$. By separately considering the cases $k > 1$ and $k < 1$, express

$2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)$ in the form $R\cos(\theta+\alpha)$, where $R(>0)$ in terms of k, and $\alpha\,(0<\alpha<2\pi)$ are real constants to be determined.

Hence, solve $2k\cos\left(\theta+\frac{\pi}{3}\right)+2\sin\left(\theta-\frac{\pi}{6}\right)=|k-1|$.

(*b*) In the quadrilateral $ABCD$ shown in the figure $AB = 2p$, $CD = 4p$, $A\hat{C}B = \frac{\pi}{6}$ and $A\hat{B}C = A\hat{C}D = \alpha$. Show that $AD^2 = 16p^2(\sin^2\alpha - \sin 2\alpha + 1)$.

Hence, show that if $AD = 4p$, then $\alpha = \tan^{-1}(2)$.



(*c*) Solve, $\tan^{-1}(\ln x^{\frac{2}{3}})+\tan^{-1}(\ln x)+\tan^{-1}(\ln x^2)=\frac{\pi}{2}$ for $x>1$.

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අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2022(2023)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2022(2023)
General Certificate of Education (Adv. Level) Examination, 2022(2023)

සංයුක්ත ගණිතය II
இணைந்த கணிதம் II
Combined Mathematics II

10 E II

Part B

* Answer five questions only.

(In this question paper, g denotes the acceleration due to gravity.)

11. (*a*) A car P that begins its journey from rest from a point O on a straight horizontal road travels with a constant acceleration $2f$ m s^{-2} up to a point A on that road, where $OA = a$ m. It maintains the velocity attained at A throughout its remaining journey. At the instant when car P reaches the point A, another car Q begins its journey, along the same road in the same direction, from rest at the point O and moves with a constant acceleration f m s^{-2}. Sketch the velocity-time graphs for the motion of P and Q in the same diagram.

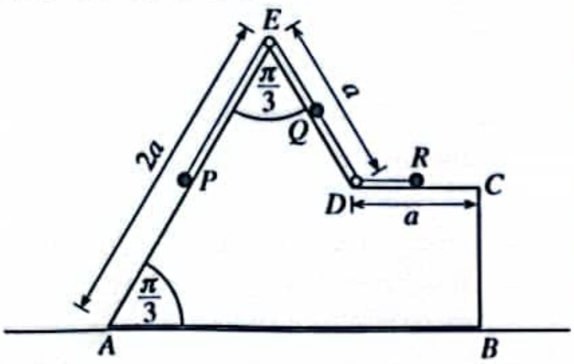
Hence, show that the time taken by Q to the instant when the velocities of P and Q are equal is $2\sqrt{\frac{a}{f}}$ s.

Now, let $a = 50$ and $f = 2$, and let B be the point on the road at which the car Q passes the car P. Show that $AB = 50\left(5+2\sqrt{6}\right)$ m.

(*b*) A ship P is sailing due South with a uniform speed 60 m s^{-1} relative to earth and a ship Q is sailing due East with a uniform speed $30\sqrt{3}$ m s^{-1} relative to earth. A third ship R appears to move in the direction 30° North of East when it is observed from P and ship R appears to move due South when it is observed from Q. Show that the ship R moves in the direction 30° South of East with a speed 60 m s^{-1} relative to earth.

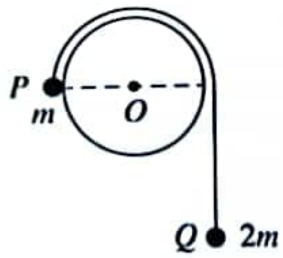
Suppose that initially the ship R is located 24 km away from P in a direction 60° South of West and 6 km away from Q in due West. Show that the distance between Q and R is 12 km, when P and R are the shortest distance apart.

12. (*a*) The vertical cross-section $ABCDE$ through the centre of gravity of a smooth uniform block of mass $4m$ is shown in the figure. The face containing AB is placed on a smooth horizontal floor. Also, AE and ED are the lines of greatest slope of the faces containing them. $AE = 2a$, $ED = a$, $DC = a$ and $E\hat{A}B = A\hat{E}D = \frac{\pi}{3}$. Three particles P, Q and R of masses $3m$, $2m$ and m, respectively, are placed at the mid-points of AE, ED and DC. The particles P and Q are attached to the ends of a light inextensible string passing over a smooth light small pulley fixed to the block at E, and the particles Q and R are attached to the ends of another light inextensible string passing through a smooth light small ring fixed to the block at D. Strings are taut in the position shown in the diagram and the system is released from rest from this position. Obtain equations sufficient to determine the time taken for the particle Q to reach E.



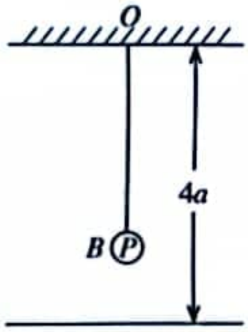
[see page eight

(*b*) A cylinder of radius a is fixed with its axis horizontal and the adjoining figure shows a vertical cross-section of the cylinder perpendicular to its axis. Two particles P and Q of masses m and $2m$, respectively connected by a light inextensible string are held with the string taut and OP horizontal in the position as shown in the figure and released from rest. Assuming that the particle Q moves vertically downwards, show that the speed v of the particle P when $\overrightarrow{OP}$ has turned through an angle θ $(0 \leq \theta \leq \frac{\pi}{6})$ is given by $v^2 = \frac{2ga}{3}(2\theta - \sin\theta)$.



The string is cut when $\theta = \frac{\pi}{6}$ and it is given that the particle P moving on the cylinder comes to instantaneous rest before it reaches the highest point of the cylinder. In the subsequent motion, find the speed of P when it is at a distance a vertically below its initial position.

13. One end of a light elastic string of natural length $2a$ and modulus of elasticity $2mg$ is attached to a fixed point O which is at a distance of $4a$ above a smooth horizontal floor, and the other end to a particle P of mass m. The particle P hangs in equilibrium at B. Show that the extension of the string is a.



Now, the particle P is given an impulse of mv vertically downwards.

Show that the equation of motion of P is $\ddot{x} + \omega^2 x = 0$ where $\omega = \sqrt{\frac{g}{a}}$ and $BP = x$.

Using the formula $\dot{x}^2 = \omega^2(c^2 - x^2)$, where c is the amplitude, show that if $v > \sqrt{ag}$, P hits the floor.

Now, suppose that $v = 3\sqrt{ag}$.

Find the velocity with which P hits the floor.

The coefficient of restitution between P and the floor is e. If $e < \frac{1}{\sqrt{2}}$, show that the particle P will not reach O.

If it is given that $e = \frac{1}{2}$, find the velocity of P when the string becomes slack for the first time.

Find the total time taken by P to come to instantaneous rest for the first time, from the instant it was given the impulse at B.

14. (*a*) Let the position vectors of four points A, B, C and D be $\mathbf{a}$, $\mathbf{b}$, $3\mathbf{a}$ and $4\mathbf{b}$, respectively with respect to a fixed origin O, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. E is the point of intersection of AD and BC. Using the triangle law of addition for the triangle OAE, show that $\overrightarrow{OE} = \mathbf{a} + \lambda(4\mathbf{b} - \mathbf{a})$ for $\lambda \in \mathbb{R}$.

Similarly, show also that $\overrightarrow{OE} = \mathbf{b} + \mu(3\mathbf{a} - \mathbf{b})$ for $\mu \in \mathbb{R}$.

Hence, show that $\overrightarrow{OE} = \frac{1}{11}(9\mathbf{a} + 8\mathbf{b})$.

(*b*) Three forces $\alpha\mathbf{i} + 2\mathbf{j}$, $-3\mathbf{i} + \beta\mathbf{j}$ and $\mathbf{i} + 5\mathbf{j}$ act through the points with position vectors $\mathbf{i} + \mathbf{j}$, $3\mathbf{i} + \mathbf{j}$ and $2\mathbf{i} + 2\mathbf{j}$, respectively, where $\alpha, \beta \in \mathbb{R}$. It is given that this system of forces is equivalent to a couple. Find the values of α and β, and the moment of this couple.

Now, a new force $3\gamma\mathbf{i} + 4\gamma\mathbf{j}$ acting through the origin O is added to the above system of forces, where $\gamma > 0$. Show that the new system consisting of 4 forces is equivalent to a resultant force and, find its magnitude, direction and the equation of its line of action.

Next, it is given that when a force $p\mathbf{i} + q\mathbf{j}$ acting through the point with position vector $2\mathbf{i} + 3\mathbf{j}$ is added, this system consisting of 5 forces is in equilibrium. Find the values of γ, p and q.

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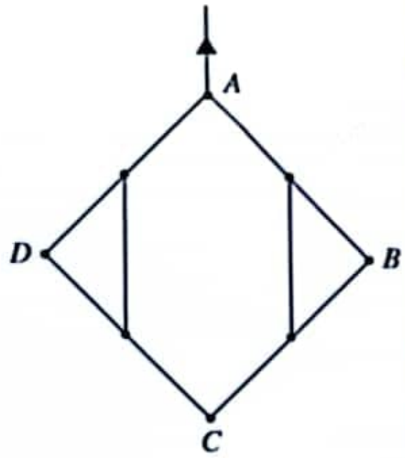
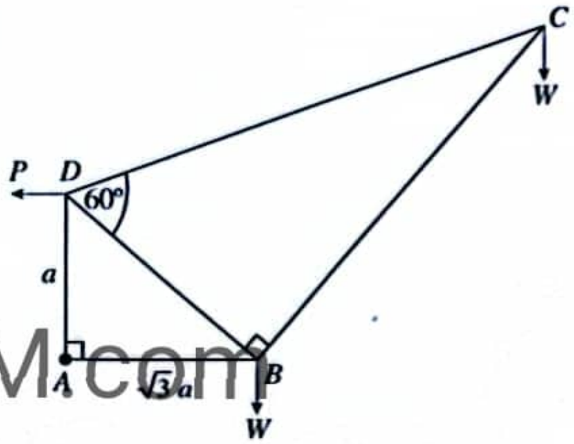
15.(*a*) Four uniform rods AB, BC, CD and DA each of length $2a$ and weight W are smoothly jointed at their ends A, B, C and D. The midpoints of AB and BC are joined by a light inextensible string of length a. Similarly, midpoints of AD and DC are also joined by a light inextensible string of length a. The system is suspended in a vertical plane from the point A and stays in equilibrium as shown in the figure. Find the tensions in the strings and the reaction exerted on AB by BC at the joint B.



(*b*) The framework shown in the figure consists of five light rods AB, BC, CD, DA and DB smoothly jointed at their ends. It is given that $AD = a$, $AB = \sqrt{3}\,a$, $B\hat{A}D = 90^\circ$, $C\hat{B}D = 90^\circ$ and $B\hat{D}C = 60^\circ$. At each of the joints B and C, a load W is suspended and the framework is smoothly hinged at A to a fixed point and kept in equilibrium in a vertical plane with AB horizontal by a horizontal force P applied to it at the joint D.

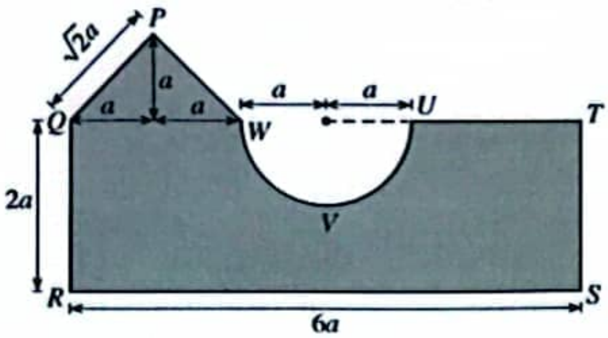


(i) Find the value of P.

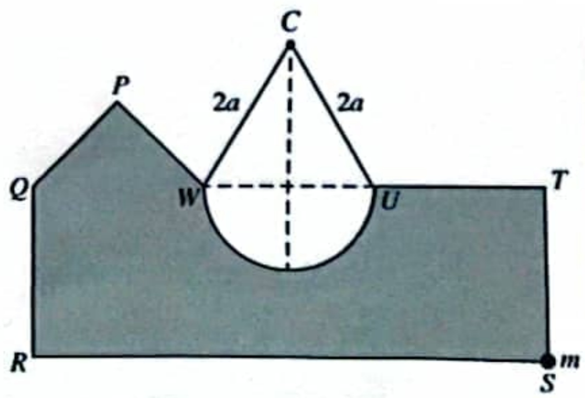
(ii) Draw the stress diagram using Bow's notation for the joints C, B and D. Hence, find the stresses in the rods, stating whether they are tensions or thrusts.

16. Show that the centre of mass of a uniform semi-circular lamina of radius r and centre O is at a distance $\frac{4r}{3\pi}$ from O.

A plane lamina is made from a uniform thin sheet metal of surface density σ by removing a semi-circle of radius a from the rectangle $QRST$ and by adding an isosceles triangle PQW with equal side-lengths $\sqrt{2}a$ to it, as shown in the adjoining figure. $QR = 2a$, $RS = 6a$ and $QW = 2a$. The centre of mass of this lamina lies at a distance $\bar{x}$ from QR and $\bar{y}$ from RS. Show that $\bar{x} = \frac{(74-3\pi)a}{(26-\pi)}$ and $\bar{y} = \frac{2(15-\pi)a}{(26-\pi)}$.



The lamina with a particle of mass m fixed to it at S, hangs in equilibrium by a light inextensible string of length $4a$ whose ends are attached to U and W and passing over a small smooth fixed peg C with side RS horizontal as shown in the figure. Find the value of m and the tension of the string in terms of a and σ.



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17.(*a*) Four identical boxes B_1, B_2, B_3 and B_4, each contains 4 pens which are identical in all aspects except for their colour. Each box B_k contains k red pens and $4-k$ black pens for $k = 1, 2, 3, 4$. One of the four boxes is chosen at random and 2 pens are drawn from that box. Find the probability that

(i) the two pens drawn are red,

(ii) the pens are drawn from box B_4, given that the two pens drawn are red.

(*b*) The data sets $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$ have the same mean, and their standard deviations are σ_x and σ_y, respectively. Show that the variance of the combined data set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ is given by $\dfrac{n\sigma_x^2 + m\sigma_y^2}{n+m}$.

Diameters of bolts produced at a factory is summarised in the following table:

Diameter (mm)	Number of bolts (in thousands)
2 – 6	2
6 – 10	5
10 – 14	8
14 – 18	4
18 – 22	1

Estimate the mean, the median and the variance of the distribution given above.

The diameters of another 40 000 bolts produced by a neighboring factory has the same mean, while the variance is 22.53 mm^2. Estimate the combined variance of the diameters of the bolts produced by both factories.

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